

Trans-Planckian redshifts and the substance of the space-time river

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Trans-Planckian redshifts in cosmology and outside black holes may provide windows on a hypothetical short distance cutoff on the fundamental degrees of freedom. In cosmology, such a cutoff seems to require a growing Hilbert space, but for black holes, Unruh's sonic analogy has given rise to both field theoretic and lattice models demonstrating how such a cutoff in a fixed Hilbert space might be compatible with a low energy effective quantum field theory of the Hawking effect. In the lattice case, the outgoing modes arise via a Bloch oscillation from ingoing modes. A short distance cutoff on degrees of freedom is incompatible with local Lorentz invariance, but may nevertheless be compatible with general covariance if the preferred frame is defined non-locally by the cosmological background. Pursuing these ideas in a different direction, condensed matter analogs may eventually allow for laboratory observations of the Hawking effect. This paper introduces and gives a fairly complete but brief review of the work that has been done in these areas, and tries to point the way to some future directions.

§1. Introduction

It was inspiring to be in Kyoto with the purpose of looking forward toward the next century of gravitational physics. The present century gave us general relativity, with its profound and beautiful understanding of gravitation. In this view, we can think of spacetime as a river, and of gravitation as the inhomogeneity of the flow. At the YKIS99 meeting we heard about vortices, waves, bifurcations, and stones in the river. The subject of my talk was a question about the substance of the river itself: the ancient question whether space and time are continuous or discrete.

The ultraviolet divergences of quantum field theory and the infinite curvature singularities of general relativity call for a fundamental short distance cutoff of some kind. Perhaps spacetime is locally discrete, or perhaps locality itself is not valid, as string theory suggests. But to learn something about the physics of the cutoff we must find a way to access that remote territory.

The modern viewpoint¹⁾ is that quantum field theory is an effective description of collective degrees of freedom of an underlying medium whose nature remains unknown, and need not be known, in order to do physics below some cutoff momentum scale. However, there are two familiar settings in which the usual separation between long and short distance scales breaks down, and the short distance physics is, potentially, unmasked. These are the expansion of the universe, and the event horizon of a black hole. In both cases there is a tremendous, trans-Planckian, redshift.

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The redshift at a black hole horizon means that the low energy effective field theory is not self-contained, since low energy outgoing modes evolve from degrees of freedom above the cutoff. The same is true in cosmology, since the expansion of the universe redshifts degrees of freedom from above to below the effective field theory cutoff. In cosmology, the puzzle seems to be even deeper than for a black hole, since the number of degrees of freedom must grow as the universe expands if there is a fundamental short distance cutoff. In the black hole problem, by contrast, there is a time translation symmetry of the black hole spacetime, and translational symmetry is broken, which may allow for a solution to the puzzle not involving a time dependence of the number of underlying degrees of freedom.

It would seem natural to address the trans-Planckian redshift puzzle in string theory, since in this theory there are no degrees of freedom localizable below the string scale. Unfortunately, the remarkable success in understanding the process of Hawking radiation of closed string states by near-extremal D-branes²⁾ seems to be of no help in this endeavor, since it concerns processes at weak coupling in a flat background spacetime. The challenge is to account for the evolution of the ingoing degrees of freedom into the outgoing modes (and therefore the Hawking radiation) in the presence of a horizon. To address this challenge would require an understanding of localized observables in a black hole background at strong coupling, an understanding not presently available in string theory.

The situation may turn out to be better in the framework of the “holographic” AdS/CFT duality³⁾, which is an outgrowth of D-brane theory. If true, this duality would solve both the UV divergence and curvature singularity problems, the former because the CFT is finite, and the latter because the CFT lives on a non-dynamical background spacetime and has no gravity in it! Although the description of localized bulk observables in terms of observables of the dual boundary CFT remains an elusive quest, with no sign of rapid progress in sight, perhaps the existence of an in-out mapping can be understood without solving this problem. I leave such a possibility for future work, and concentrate in this article on other approaches to understanding the trans-Planckian puzzle.

Rather than reproducing the content of my talk at YKIS99, most of which is anyway covered more completely in published articles^{4), 5)}, I have taken the opportunity of this contribution to step back and present a broader view of the whole subject. The underlying ideas and motivations are discussed here, and I have made an effort to review all of the related work of which I am aware. I have also tried to look forward and discuss possibilities for future work that might be worthwhile and important.

The remainder of this paper is organized as follows. The cosmological and black hole trans-Planckian redshift puzzles are discussed in section 2. In section 3 the sonic analogy is introduced, and the dispersive field theory models to which it gave rise are discussed. Section 4 is devoted to lattice models, and section 5 discusses the various condensed matter analogs that have been proposed so far. Finally, section 6 addresses the issue of Lorentz non-invariance.

§2. Trans-Planckian redshift puzzles

2.1. Cosmological redshift

In a pioneering paper written in 1985 (in a field that does not yet exist), Weiss⁶⁾ pointed out serious obstacles for the problem of hamiltonian lattice quantum field theory in an expanding universe, which I will now summarize. If the lattice points are co-moving with the cosmological expansion, their proper separation grows in time, which leads to difficulties to be discussed momentarily. One can instead place the lattice points on trajectories that maintain their proper separation as follows.

In a spatially flat Robertson-Walker spacetime, for example, with line element $ds^2 = dt^2 - a(t)^2(dr^2 + r^2d\Omega^2)$, the new coordinate $\rho = a(t)r$ casts the line element in the form $ds^2 = (1 - H^2\rho^2)dt^2 + 2H\rho d\rho dt - d\rho^2 - \rho^2d\Omega^2$, where $H = d(\ln a)/dt$ is generally a function of time. If lattice points were to sit at fixed (ρ, θ, ϕ) , their proper separation at fixed t would be constant. However, since the vector $\partial/\partial t$ is spacelike for $\rho > H^{-1}$, the Hamiltonian generating this “time-translation” would be unbounded below. Weiss suggested this problem could be avoided with a boundary condition at $\rho = H^{-1}$, but noted that this does not look like a promising approach, since the boundary condition is in general time-dependent. The nature of the surface $\rho = H^{-1}$ depends on the function $a(t)$.⁷⁾ For $a(t) = t^n$, the surface is spacelike if $n < 1/2$, outgoing lightlike if $n = 1/2$ and timelike if $n > 1/2$. In any case, information flows into the region $\rho < H^{-1}$ from the boundary, and one doesn’t know what this information should be without solving the dynamics outside the boundary. In the spacelike case one could impose the data as an “initial” condition, however since the boundary is not a surface of constant t one would not know what condition is appropriate without solving the dynamics outside the boundary.

In the case of deSitter space where H a constant, the surface $\rho = H^{-1}$ is ingoing lightlike, and is in fact just the deSitter horizon. Information cannot cross this horizon from outside to inside, so it plays the same role for the inside as a black hole horizon plays for the exterior of a black hole. But even in this case there is a difficulty, since the acceleration of the lattice points at fixed (ρ, θ, ϕ) diverges as the horizon is approached. According to the discussion in section 4 on lattice black holes, I would expect this infinite acceleration to lead to unphysical behavior of the lattice theory.

If the lattice points are instead co-moving with the expansion, their proper separation grows in time. To maintain small enough lattice spacing for a calculation spanning several e-foldings of the scale factor, one must begin with an exponentially small spacing. Moreover, since the density of states is then time dependent, the bare parameters of the theory must be functions of the cosmic time in order to keep the renormalized parameters fixed at a fixed proper scale. It would seem terribly impractical to keep track of all the degrees of freedom and continuously adjust the bare couplings so that the renormalized couplings are constant.

Much more natural would be to somehow add lattice points as the universe expands, to maintain a constant proper density of points. This is not straightforward however, since it is not unitary evolution on a fixed Hilbert space. Nevertheless, if

there is a fundamental short distance cutoff, some such generalization of quantum theory seems unavoidable. It also seems desirable, because it might neatly account for the low initial entropy of the universe, since the actual number of degrees of freedom would have been very small at the beginning.

The idea of a growing cosmological Hilbert space has been discussed in a few places I am aware of^{(8), (9), (10)}, and a proposal for how it might emerge from quantum gravity is discussed in⁽¹⁰⁾. The idea of a changing Hilbert space in general is natural in the consistent histories formulation of quantum mechanics, as has been emphasized by Hartle⁽¹¹⁾. The unitary inequivalence of the Hilbert space at different times for a quantum field in curved spacetime has recently been formulated by Anastopoulos⁽¹²⁾ in terms of a time-dependent Hilbert space using the histories formalism. It also seems that the algebraic formulation of quantum field theory⁽¹³⁾ has room for what would amount to a time dependent Hilbert space.

A growing Hilbert space was introduced for practical reasons in a continuum-based calculation of Ramsey and Hu⁽¹⁴⁾. They numerically evolved the scale factor coupled to a self-interacting quantum scalar field in inflationary semiclassical cosmology, using a $1/N$ approximation. For economy of computation, at any given time in the evolution only modes up to some fixed large proper cutoff wavevector were included, the higher modes being physically irrelevant. This meant that, as the universe expanded, entirely new modes entered the Hilbert space of the calculation. These modes were inserted in their instantaneous adiabatic vacuum state. This sort of scheme was previously suggested as an effective description of cosmology with a growing Hilbert space⁽¹⁵⁾.

2.2. Black hole horizon redshift

If the short wavelength outgoing modes near the event horizon of a black hole are in their ground state (as defined by the time of an infalling observer), then these modes will be occupied at the Hawking temperature at infinity. By “short wavelength” and “near” here I mean compared with the size of the black hole, as measured by an infalling observer. Whatever the details of the theory at ultrashort distances, as long as this outgoing vacuum condition is met, the Hawking effect will occur.⁽¹⁶⁾

The question that interests me, however, is by what mechanism can a theory with a cutoff produce these outgoing modes at all? If they can be accounted for, the fact that they are delivered in their ground state would be an important but presumably straightforward consequence of the fact that the vacuum at short distances—whatever it is—remains unexcited as it propagates in the black hole background, which is characterized by much longer distance and time scales.

If you are more practically than philosophically inclined, consider the problem the following way. Suppose you want to numerically calculate what comes out of an evaporating black hole, in an interacting quantum field theory, using a lattice formulation. How can the lattice theory with, say, Planck scale lattice spacing, possibly produce the outgoing modes, when it clearly cannot “store” them in a trans-Planckian reservoir at the horizon?

When I first started thinking about this problem I suspected that, as in the

cosmological case, it would be necessary to generate new states in the Hilbert space, corresponding to the outgoing modes. However, as stated earlier, the black hole is stationary and breaks translational symmetry, so the analogy with cosmology is not perfect. Indeed, it seems that there is a way to produce the outgoing modes within the confines of a fixed Hilbert space.

§3. Sonic analogy and dispersive field theory models

A condensed matter analogy, Unruh’s sonic black hole analogy^{17), 18)}, led the way to a possible resolution of the puzzle of the origin of the outgoing black hole modes.^{*)} I say “a possible” resolution rather than “the” resolution, since the proposed scenario requires a kind of modification of fundamental short distance physics which has not been incorporated into any complete theory, let alone been confirmed experimentally.

3.1. *Sonic analogy*

The idea of Unruh’s analogy is that when fluid flows faster than the speed of sound somewhere in an inhomogeneous flow, a sonic horizon appears which is in many ways analogous to a black hole horizon. The sound field—perturbations of the fluid flow—behaves in fact precisely like a massless relativistic field in a curved background spacetime determined by the flow parameters, provided the flow is irrotational, barotropic and inviscid.^{17), 21), 22)} For our present purposes the essential point is independent of these details, and is just the fact that the sonic horizon is an infinite redshift surface for sound waves. This poses a “trans-Bohrian” puzzle, since a real physical fluid cannot support sound of arbitrarily short wavelength. In particular, at wavelengths smaller than the intermolecular spacing the effective field theory of sound becomes invalid and the density of states drops to zero. Nevertheless, in the presence of such a sonic horizon there must indeed be outgoing sound modes, for these are just some of the collective degrees of freedom of the fluid that must exist.

So how does a real fluid manage to produce the outgoing modes at a sonic horizon? Since the sound speed is the top speed for excitations of the fluid, modes cannot propagate from inside to outside the horizon. The only possibility that conserves the number of degrees of freedom of the fluid is therefore that the outgoing modes come from ingoing degrees of freedom that are somehow turned back at the horizon. (There can be no talk of creating degrees of freedom in a fluid, which is surely in principle a self-contained quantum system.) Though this initially sounds rather strange, especially given that a horizon is normally thought to swallow everything that approaches it, it turns out to be an example of a general phenomenon that occurs for dispersive waves in an inhomogeneous medium.

Clearly modes cannot be “reflected” from the horizon, since there is no sharp interface at which reflection could take place. Rather, the reversal of group velocity must happen continuously, as a result of smooth evolution from one branch of

^{*)} A very different suggestion for how to avoid the need for fundamental high frequency degrees of freedom, using “superoscillations”, is discussed in Refs.¹⁹⁾ and ²⁰⁾.

the dispersion relation to another. The dispersion relation relating frequency ω to wavevector k for an atomic fluid has the form $\omega = c_s k$ for wavelengths long compared to the interatomic spacing, but as the wavevector grows this is modified and the group velocity drops lower than the long wavelength speed of sound c_s . In superfluid Helium-4, for example, the group velocity actually drops to zero and then reverses sign before the so-called roton minimum is reached.

Due to this drop of group velocity, an outgoing mode traced backwards in time in the WKB approximation will asymptotically approach a point *outside* the sonic horizon, where the group velocity and flow velocity are equal and opposite.¹⁵⁾ What happens at this point is that the group velocity continues to drop, and the wavepacket reverses direction and propagates back away from the horizon.¹⁸⁾ In other words, now viewing the process forward in time, an “outgoing” short wavelength mode with low group velocity is dragged in toward the horizon by the faster fluid flow. As this happens the wavevector decreases and the group velocity increases, eventually reaching and then exceeding the flow velocity, at which point the wavepacket begins propagating back out away from the horizon.

3.2. Dispersive field theory models

It is not necessary to solve the many-body problem of a real atomic fluid in order to explore this phenomenon. Unruh¹⁸⁾ studied the scenario just described by numerically solving the wave equation for a 1+1 dimensional free field theory, with higher spatial derivative terms designed to produce a dispersion relation of the form

$$\omega = k_0 \left(\tanh(k/k_0)^n \right)^{1/n} \quad (3.1)$$

in the co-moving frame. The wavevector k_0 sets the scale at which the deviations from the massless wave equation become important. Unruh confirmed the reversal of group velocity, and found that in the process there is some conversion from the positive to the negative frequency branch of the dispersion relation, in just the right amount to yield the Hawking rate for particle production when the calculation is interpreted quantum mechanically.

Unruh’s model can be interpreted as a field theory in a two dimensional black hole spacetime, without any reference to fluid flow.^{*)} As such, it becomes a model of how quantum fields may behave in a black hole background if there is modified dispersion at high wavevectors. The modified dispersion relation is not Lorentz invariant, so one must specify the local frame in which the dispersion is specified. The spacetime analog of the co-moving fluid frame is the free-fall frame of the black

*) The corresponding spacetime is geodesically incomplete in somewhat the same way the Eddington-Finkelstein coordinates are incomplete. (For the details in a slightly different setting see Ref.²³⁾.) This is curious, since from the Newtonian point of view the fluid it must of course be *physically* complete. The Lorentzian incompleteness is physically irrelevant since, as a wave propagates towards the edge of the spacetime, it blueshifts beyond the linear, Lorentz-invariant part of the dispersion relation. Thus, for example, a wave can not fall off the spacetime running backwards in time along the horizon since it will first blueshift and, in the subluminal case, undergo a reversal of group velocity which takes it back away from the horizon. In the superluminal case it will cross the horizon.

hole, and this frame has been adopted in most of the work to date. As discussed in Section 6, the results are independent of the choice of preferred frame as long as it is not too accelerated or boosted relative to the black hole.

Subsequent to Unruh’s calculation much work has been done to understand, confirm, and extend the result. Here I will briefly review this work. First, the result was explained by Brout et. al.²⁴⁾ using a WKB analysis, in which they computed the Bogoliubov coefficients at leading order. They also showed how the trajectories of the wavepacket and its negative energy “partner” can easily be found from the geometric optics limit of the modified wave equation.

The result was confirmed to very high precision by Corley and Jacobson²⁵⁾, by exploiting the stationarity to reduce the problem to one of (numerically) solving the ODE’s for modes of fixed frequency satisfying the appropriate (damped) boundary condition inside the horizon. In order to keep the order of the ODE’s from going higher than four derivatives, the dispersion relation

$$\omega^2 = k^2 - k^4/k_0^2 \quad (3.2)$$

was adopted, the idea being that the k^4 term is just the lowest order term in the derivative expansion of a generic (subluminal) dispersion relation. This work also revealed a new mechanism of particle creation, in addition to the Hawking radiation, in which particles are created by static curvature which however is not static in the frame in which the dispersion relation is specified. Corley²⁶⁾ further studied this new type of particle creation in settings where there is no black hole, and explored the modification of the Hawking spectrum in the limit where the Hawking temperature approaches or exceeds the scale k_0 at which the dispersion relation is modified.

Corley also developed²⁷⁾ the technology for analytical calculations of the Hawking effect in dispersive field theory models, using the method of matched asymptotic expansions. In this method, one solves the ODE for the mode functions by the method of Laplace transform in a neighborhood of the horizon, and matches this solution to the WKB modes away from the horizon. This technology has previously been well developed in the context of plasma wave theory²⁸⁾, where the analogs of the wave phenomena we are discussing go by the name of “mode conversion”. Using this method, Corley reproduced the leading order Hawking result, and treated also the case of superluminal propagation at high wavevectors, which is relevant to some condensed matter models (see below). He showed that in the presence of superluminal high frequency dispersion the Hawking effect would also be recovered at leading order, provided the modes approaching the horizon from the inside are in their ground state.

Corley’s methods have recently been extended by both Himemoto and Tanaka²⁹⁾ and by Saida and Sakagami³⁰⁾, to find the leading order deviations from the thermal Hawking spectrum. In Ref.³⁰⁾ a precise form for the leading deviation is found, for frequencies in the range $\kappa < \omega < \kappa(k_0/\kappa)^{2/5}$, where κ is the surface gravity. This deviation is of order $\omega^3/\kappa k_0^2$, in agreement with the results of in Ref.²⁹⁾ which show that around the peak of the Hawking spectrum, the corrections are generically of order κ^2/k_0^2 . Both of these papers confirm this prediction with numerical calculations (although the sign of the corrections found numerically in Ref.³⁰⁾ does not match the

analytically computed correction). This result disagrees with the earlier numerical results of Ref. ²⁵⁾, which showed an even smaller deviation from the thermal result. As shown in both these papers, however, this discrepancy is explained by the fact that the particular black hole metric adopted in Ref. ²⁵⁾ was not generic in its form near the horizon, and produced abnormally small deviations.

The analysis of ²⁷⁾ was applied by Corley and Jacobson ³¹⁾ to the case, motivated by condensed matter models, where a field with superluminal high frequency dispersion propagates in a black hole background with both an inner horizon and an outer horizon. In this case it was found that the negative energy partners of Hawking quanta bounce off the inner horizon, return to the outer horizon, and stimulate more Hawking radiation if the field is bosonic or suppress it if the field is fermionic. This process leads to exponential growth or damping of the radiated flux and correlations among the quanta emitted at different times, unlike in the usual Hawking effect.

An intriguing question that remains open in these dispersive models is the behavior of Hawking radiation in the ultra low frequency limit. There is some reason to suspect a strong deviation from the thermal Hawking spectrum in this limit. The WKB wavevector k_{tp} at the turning point is of order ⁴⁾ $k_{\text{tp}} \sim \omega^{1/3} k_0^{2/3}$, which is smaller than the surface gravity κ when $\omega < \kappa^3/k_0^2$, whereas the usual approximation underlying the derivation of the Hawking spectrum assumes on the contrary that the wavevector near the horizon is much larger than κ . On the other hand, perhaps conditions at the turning point are irrelevant. In fact, at least for frequencies of order κ or greater, the Hawking spectrum seems to be determined by the behavior of the waves as they tunnel across the horizon, as evidenced by the fact that the thermal spectrum at the Hawking temperature is recovered to high precision even when the turning point recedes significantly from the horizon. ^{25), 5)}. Whether this ‘horizon dominance’ persists in the ultra low frequency regime is not obvious. It seems worthwhile to settle this question, as it would be fascinating and important if a modification of the theory at the high wavevector scale k_0 were to have an effect on the Hawking spectrum at ultra low frequencies.

3.3. *Revenge of the trans-Planckian modes*

As explained above in Section 3.1, the mechanism of producing the outgoing modes is a reversal of group velocity of high wavevector (of order k_0) modes near the horizon, brought about by the nonlinearity of the dispersion relation $\omega(k)$. This avoids the need to draw upon a reservoir of trans-Planckian modes at the horizon. However, it does not fully solve the problem. If we ask where these ingoing, short wavelength modes come from, there is no satisfactory answer. In Unruh’s model, with the tanh dispersion relation (3.1), there is no cutoff at high wavevectors, and the group velocity in the co-moving frame drops to zero as $k \rightarrow \infty$. This means that as the ingoing mode is followed further out backwards in time, it continues to blueshift without bound as the velocity of the co-moving frame (relative to the static frame) drops to zero. Thus the need for an infinite density of states has not been eliminated, it has only been pushed away from the horizon out to infinity.

In the dispersive models with the quartic dispersion relation (3.2), the problem is different. As the ingoing mode is followed backward in time, the wavevector runs

out to the end of the spectrum at $|k| = k_0$. (This happens to the positive frequency part only asymptotically where the co-moving velocity drops to zero, however the negative frequency part of the mode encounters the end of the spectrum at non-zero co-moving velocity.²⁵⁾) At that point the model is ill-behaved, since for $|k| > k_0$ the frequency is imaginary. Of course the quartic dispersion relation was never intended as a model for a fundamental theory, but only as a first order correction, so it is not disturbing that it does not make sense when pushed to sufficiently high wavevectors.

Another way to see that it is impossible to sensibly produce the outgoing modes in these models is to note that the Killing frequency is conserved since the background is static. Thus the ingoing mode must have the same frequency as the outgoing mode. However, this implies that there is no Hawking radiation, since in the absence of mixing between positive and negative frequencies the in-vacuum evolves to the out-vacuum.

In studying these models, this problem was avoided simply by imposing the ingoing vacuum boundary condition at non-zero co-moving velocity, never taking the asymptotic velocity to zero. In principle, however, the problem means that these models have not succeeded in providing a fully viable mechanism for producing the outgoing black hole modes.

§4. Lattice models

A lattice provides a simple model for imposing a physically sensible short distance cutoff, one which is more like the atomic fluid of Unruh's sonic model than are the continuum-based dispersive models. In a lattice model, space is discretized, and a linear field takes values only at the lattice points, each of which has a continuous worldline in spacetime. The discrete theory is self-contained and unitary, so there can be no physically pathological behavior at high lattice wavevectors.

The lattice succeeds in producing the outgoing modes since, on a lattice of spacing δ , the dispersion relation is sinusoidal, wavevectors are identified modulo $2\pi/\delta$, and continuous passage from left-moving to right-moving wavepacket is possible. Let me now explain this. The details are described in Refs.^{4), 5)}.

It might seem at first that the most natural choice would be to preserve the time translation symmetry of the spacetime with a static lattice whose points follow accelerated worldlines. On a static lattice, however, the Killing frequency is conserved, so outgoing modes arise from ingoing modes with the same frequency, and there is no Hawking radiation. The in-vacuum therefore evolves to a singular state at the horizon. (The equilibrium (Hartle-Hawking) state at the Hawking temperature would however presumably be regular at the horizon, hence could be modeled on a static lattice. This thermal equilibrium state could also be modeled on a static lattice in a Euclidean black hole spacetime³²⁾.) Moreover, inside the horizon the Killing field is spacelike, so the worldlines of static lattice points would be spacelike, which would make the lattice theory sick if the inside were not omitted. Finally, a static lattice is unnatural from the viewpoint of the fluid model, wherein the atoms flow across the horizon.

If the lattice points are instead falling, it is still possible to preserve a discrete

remnant of time translation symmetry. A single falling worldline can be repeatedly translated in Killing time by a discrete amount, building up a lattice. If the lattice points are asymptotically at rest at infinity however, their spacing will go to zero at infinity, so there is no fixed short distance cutoff.^{*)}

We thus insist that the lattice spacing asymptotically approaches a fixed constant *and* that the lattice points are at rest at infinity and fall freely into the black hole. In this case, the lattice cannot have even a discrete time translation symmetry, since there is a gradual spreading of the lattice points as they fall toward the horizon. The time scale of this spreading is of order $1/\kappa$ where κ is the surface gravity. This time dependence of the lattice is invisible to long wavelength modes, which sense only the stationary background metric of the black hole, but it is quite apparent to modes with wavelengths of order the lattice spacing.

On such a lattice the long wavelength outgoing modes come from short wavelength ingoing modes via a process closely analogous to the Bloch oscillation of an accelerated electron in a crystal. Bloch oscillations occur when the acceleration acts long enough for the momentum to grow to the scale of the lattice wavevector. Due to the sinusoidal nature of the dispersion relation for the electron in the lattice, the group velocity drops, and changes sign when the momentum reaches π/δ , where δ is the lattice spacing. An accelerated electron would therefore oscillate back and forth (were the motion not dissipated by coupling to other lattice degrees of freedom such as phonons).

On a falling lattice in a black hole background, something similar happens. In a freely falling, Gaussian normal coordinate system, the 1+1 dimensional black hole metric takes the form

$$ds^2 = dt^2 - a(z, t)dz^2, \quad (4.1)$$

where the “scale factor” $a(z, t)$ goes to unity at $t = 0$ and as $z \rightarrow \infty$. If the z -coordinate is discretized as $z_n = n\delta$, the dispersion relation for a massless scalar field mode $\exp(-i\omega t + ikz_n)$ on the lattice takes the form

$$\omega = \pm \frac{2}{a(z, t)\delta} \sin(k\delta/2). \quad (4.2)$$

This is the standard lattice dispersion relation, with the additional factor $1/a(z, t)$ coming from the black hole geometry.

Following an outgoing wavepacket backwards towards the horizon, the gravitational field blueshifts the wavevector as in the continuum. In a WKB worldline approximation, we can describe what happens backwards in time as follows. The group velocity drops below the velocity of lattice points near the horizon, so the wavepacket turns around with respect to the static coordinate and heads away from the horizon. At this stage, however, the wavepacket is still outgoing with respect to the lattice, and its WKB wavelength can still be rather long compared with the

^{*)} Hawking radiation on such a lattice was investigated analytically in ⁴⁾, by exploiting the discrete symmetry to reduce the 2d lattice wave equation to a 1d difference equation. This made it possible, with the help of the methods of ²⁷⁾, to analytically establish the leading order Hawking effect.

lattice spacing. The analog of Bloch oscillation has not yet occurred. In this part of the process the Killing frequency is conserved since the wavepacket cannot sense the time dependence of the lattice. As the wavepacket propagates backwards in time away from the horizon however, the blueshifting continues, eventually pushing the wavevector over the hump in the dispersion relation, so the group velocity changes sign with respect to the lattice as well. At this stage the wavelength is comparable to the lattice spacing, so the Killing frequency is no longer conserved. Finally, at early times far from the horizon, the ingoing wavepacket which produces the final outgoing wavepacket still has wavevector and frequency of order the lattice spacing.

This ingoing wavepacket not only has a different frequency than the outgoing wavepacket, it has negative frequency components as well. The presence of the negative frequency part is the signal of the Hawking effect. The appearance of the negative frequency part in this context can be understood as a consequence of conservation of degrees of freedom in the following hand-waving manner. As described so far, the entire outgoing branch of the dispersion curve seems to be produced from only a part of the ingoing branch,

$$(-\pi/\delta, -k_c) \rightarrow (0, \pi/\delta) \quad (4.3)$$

where k_c is some critical wavevector separating the ordinary ingoing modes that cross the horizon from the exotic ones that undergo the Bloch oscillation. This cannot be the whole story, however, since there appear to be more outgoing modes than ingoing modes which give rise to them. The resolution is that we left out the negative frequency part of the ingoing mode.

This WKB picture on the lattice, largely developed in ⁴⁾, was checked in ⁵⁾ by carrying out the exact calculation numerically using the lattice wave equation. The behavior of wavepackets described here was confirmed, and the Hawking radiation was recovered to within half a percent for a lattice spacing $\delta = 0.002/\kappa$ (which corresponds to a proper spacing $a(z, t)\delta \sim 0.08/\kappa$ where the wavepacket turns around at the horizon). The deviations from the Hawking effect that arise as $\kappa\delta$ is increased were also studied, and an interesting picture of how the wavepackets turn around at the horizon was revealed.

4.1. *Towards a non-expanding lattice model: back-reaction and dissipation?*

The falling lattice model has provided an intriguing mechanism—the Bloch oscillation—for getting an outgoing mode from an ingoing mode in a stationary background, but there is a serious flaw in the picture: the lattice is constantly expanding. In the fluid analogy, by contrast, the lattice of atoms maintains a uniform average density. In a fundamental theory we might also expect the scale of graininess of spacetime to remain fixed at, say, the Planck scale or the string scale (since presumably the graininess would *define* this scale) rather than expand. Can the falling lattice model be improved to share this feature?

An incompressible fluid maintains uniform density in an inhomogeneous flow by compressing in some directions and expanding in others, a process requiring at least two dimensions. At the atomic level such a volume-preserving flow involves erratic motions of individual atoms. One possible improvement of the lattice model is to

make a two-dimensional lattice that mimics this sort of volume preserving flow. It is not clear whether the motions of the lattice points can be slow enough to be adiabatic on the time scale of the high frequency lattice modes. If they cannot, then the time dependence of this erratic lattice background will excite the quantum vacuum.

In a fluid, however, the lattice is a part of the system, not just a fixed background. The physical ground state requirement is that the time-dependence of the flow be adiabatic for the fully coupled system. A similar comment applies in quantum gravity: surely, if in-out mode conversion is at play, the incoming high frequency modes are strongly coupled to the quantum gravitational vacuum. Ideally, therefore, we should try to find a model in which the background is not decoupled from the perturbations. The analogy with the dissipation of Bloch oscillations due to coupling to the lattice phonon modes strongly suggests that, in such a nonlinear model, the outgoing modes will arise from the time reverse of a dissipative process. The same idea was suggested from a very different point of view by Brout et al³³⁾, who examined the Hawking process in a dissipative, dispersive effective field theory derived from a modification of quantum commutators motivated by string theory. It is also consistent with the observation of Visser²²⁾ that the addition of a viscosity term to the Euler equation for a fluid results in a dispersion relation for perturbations similar to (3.2) but with an imaginary part: $\omega = (k^2 - k^4/k_0^2)^{1/2} - ik^2/k_0$. (In this case, when the dispersion comes entirely from viscosity, the imaginary correction is larger than the real one by a factor k_0/k .)

A first step in this direction might be to study a one dimensional quantum “chain” model in which the lattice points are non-relativistic point masses, coupled to each other by nearest neighbor interactions, and “falling” or propagating in a background potential (with or without periodic boundary conditions) with asymptotically non-zero velocity. The perturbations of such a chain are the phonon field, and the back reaction to the Hawking radiation is included (although the background potential is fixed). In a model like this one could presumably follow in detail the nonlinear origin of the outgoing modes and the transfer of energy from the mean flow to the thermal radiation.

The simplest such chain model would be one with harmonic nearest neighbor interactions. However, as pointed out by Unruh³⁴⁾, a harmonic chain cannot possess a horizon, since the sound velocity v_s is proportional to the inter-atomic spacing, as is the flow velocity $v(x)$ in a stationary flow (because the mass current is uniform). When the chain is stretched in a region of higher flow velocity, the ratio $v_s(x)/v(x)$ therefore remains constant, so no horizon forms. This is unfortunate, since the harmonic chain could have been solved exactly, at least asymptotically. Perhaps the idea would work with another exactly solvable model, such as the Calogero-Sutherland model³⁵⁾, which consists of nonrelativistic point masses coupled by $1/x^2$ interactions. Of course this model is exactly solvable only in isolation, not in a background potential such as needed to accelerate the chain near the horizon. Nevertheless, the asymptotic exact solution may be useful. Alternatively, perhaps no exact solution is needed at all.

§5. Condensed matter analogs

The title of Unruh's original paper¹⁷⁾ on the sonic analogy was "Experimental black-hole evaporation?" He was not only proposing a fluid model in which the effects of a short distance cutoff and quantum back-reaction to the Hawking radiation should be more understandable than in quantum gravity, he was also suggesting that these processes might be experimentally observable in the analog fluid system.

The Hawking effect is a quantum process involving an instability of the ground state due to the presence of the ergoregion behind the horizon. If a fluid model is to produce identifiable Hawking radiation, therefore, the flow should be in its quantum ground state (or at least be as cold as the Hawking temperature), rather than in an incoherent thermal state. For this reason we should contemplate setting up a horizon in a superfluid at zero temperature. The case of superfluid ^4He was initially examined in¹⁵⁾, and further discussed in³⁶⁾. It was concluded that a sonic horizon cannot be established in superflow, because the flow is unstable to roton creation at the Landau velocity which is some four times smaller than the sound velocity.

There may be other condensed matter systems, however, where a Hawking effect analog can be observed. One system that has been studied^{37), 38), 23), 39)} is the (anisotropic) A-phase of superfluid ^3He ^{40), 41)}, which has a rich spectrum of massless quasiparticle excitations. In particular, there are fermionic quasiparticles—the "dressed" helium atoms—which have gapless excitations near the gap nodes at $\vec{p} = \pm p_F \hat{l}$ on the anisotropic Fermi surface, and therefore can play the role of a massless relativistic field in a black hole analog. The unit vector \hat{l} is the direction of orbital angular momentum of the p -wave Cooper pairs and p_F is the Fermi momentum.

The velocity of fermion quasiparticles parallel to \hat{l} in $^3\text{He-A}$ is the Fermi velocity $v_F \sim 55$ m/s, while their velocity perpendicular to \hat{l} is only $c_\perp = \Delta/p_F \sim 3$ cm/s, where $\Delta \sim T_c \sim 1$ mK is the energy gap. It should be possible to set up an inhomogeneous superflow exceeding the slow speed c_\perp in a direction normal to \hat{l} , thus creating a horizon for the fermion quasiparticles. There is a catch, however, since the superflow is unstable when the speed relative to a container exceeds c_\perp .³⁸⁾ A possible way around this was suggested by Volovik³⁹⁾, who considered a thin film of $^3\text{He-A}$ flowing on a substrate of superfluid ^4He , which insulates the ^3He from contact with the container. He imagined a radial flow on a torus, such that the flow velocity near the inner radius exceeds c_\perp , producing a horizon. Theoretically this looks promising, however the Hawking temperature for a torus of size R is $T = (\hbar/2\pi)(dv/dr) \sim \hbar c_\perp/R \sim (\lambda_F/R)$ mK, where λ_F is the Fermi wavelength, which is of the order of Angstroms. Thus, even for a micron sized torus, the Hawking temperature would be only $\sim 10^{-7}$ K.

An alternative is to keep the superfluid at rest with respect to the container, but arrange for a texture in the order parameter to propagate in such a way as to create a horizon. For example, in³⁷⁾ a moving "splay soliton" is considered. This is a planar texture in which the \hat{l} vector rotates from $+\hat{x}$ to $-\hat{x}$ along the x -direction perpendicular to the soliton plane. A quasiparticle moving in the x -direction thus goes at speed v_F far from the soliton and at speed c_\perp in the core of the soliton. If the soliton is moving at a speed greater than c_\perp , the quasiparticles will not be able to keep up

with it, so an effective horizon will appear. This example turns out to be rather interesting and complicated in the effective relativistic description. The horizon has a translational velocity, making it like that of a rotating black hole rather than a static black hole. In addition, there is a strong pseudo-electromagnetic field outside the “black hole”, which would produce quasiparticles by pseudo-Schwinger pair production.⁴¹⁾ (This latter process may be the same as what produces the so-called “orbital viscosity”⁴⁰⁾ of a time-dependent texture.) The Hawking temperature also tends to be very low, and it seems likely that the Hawking effect would be masked by the pseudo-Schwinger pair production, though this has not been definitively analyzed.

In²³⁾ a simpler system was studied, that of a thin film of $^3\text{He-A}$, perhaps on a ^4He substrate, with a domain wall in which the condensate is in a different superfluid phase and across which the direction of \hat{l} , which is perpendicular to the film, flips sign. Inside the wall the group velocity of the quasiparticles goes to zero, so if the wall itself is propagating, a horizon will appear. The effective spacetime geometry of this system was studied in²³⁾, and it is a potentially interesting black hole candidate. However, the presence of the moving domain wall raises questions about the evolution of the quasiparticle vacuum that have not been addressed. Moreover, in this as in any model in which the ergoregion has a finite extent, bounded on the inside by a white hole or inner horizon, it would be necessary to understand the time scale on which the filling of the negative energy states in the ergoregion would turn off the Hawking process. In a black hole, by contrast, the negative energy states just fall into the singularity, never to be heard from again.

Turning away from $^3\text{He-A}$, some other systems have been considered as candidates for black hole analogs in condensed matter. A non-axisymmetric vortex in $^3\text{He-B}$ has gapless excitations in the states bound to the vortex core, and if the core is rotating this can lead to a black hole analog for these modes, as discussed by Kopnin and Volovik⁴²⁾. Reznik⁴³⁾ discussed a model involving a dielectric medium with spatially varying index of refraction. Hochberg and Pérez-Mercader⁴⁴⁾ developed a liquid model for black hole thermodynamics (but not for the Hawking effect). Most recently, the possibility of realizing a sonic black hole analog in a dilute Bose-Einstein condensate has been discussed.⁴⁵⁾ The condensate could possibly be made to flow in a circulating manner, with a constriction leading to a black hole/white hole horizon pair with an ergoregion in between. The dispersion relation for sound in this system is “superluminal” at large wavevectors, so the system falls into the class of models shown to be unstable to a runaway process of stimulated emission of Hawking radiation.³¹⁾ The analysis of Ref.³¹⁾ was based on a study of WKB wavepackets, which may not be a valid approximation anywhere in the regime of interest, however a numerical investigation of the linearized modes of the Gross-Pitaevskii equation appears to bear out the same conclusion.⁴⁵⁾

We must now leave the topic of condensed matter analogs with many open questions. Hopefully one day some systems in which a Hawking effect can be observed will be identified.

§6. Lorentz non-invariance

A short distance cutoff or modified dispersion relation requires a breaking of local Lorentz invariance (or perhaps a breakdown of locality itself), since the distinction between long and short wavelengths depends on the frame of reference. In the models discussed so far, this frame was taken to be the one defined by geodesics that are asymptotically at rest at infinity and fall across the horizon—the “free-fall frame”. In terms of the unit velocity 2-vector u and the Killing vector ξ , this frame is specified by the unit energy condition $u \cdot \xi = 1$. The rate of change of ξ^2 along the free-fall worldlines characterizes the time-dependence of the black-hole background seen from the point of view of the free-fall frame. At the horizon this rate takes the value -2κ , so the time scale associated with the free-fall motion is of the same order as that defined by the surface gravity κ . It is plausible, however, that as long as the choice of preferred frame does not introduce a time scale comparable to $1/k_0$, the results should not depend on this choice at leading order.

The dependence of the results on the choice of preferred frame has been studied by Himemoto and Tanaka²⁹⁾. They consider accelerated frames (which are in fact the free-fall frames of metrics related to the original metric by a static conformal factor). They find analytically that, as long as the acceleration of the frame is not too drastic, the leading deviation from the thermal spectrum occurs at order $1/k_0^2$. They also investigated numerically the case where the acceleration becomes large, so the frame is prevented more and more from falling freely, and found that the created particle flux drops significantly, by something of order unity. They conjecture that in the case of a static frame, which is infinitely accelerated at the horizon, there may be no Hawking radiation at all. A similar observation was made in the context of a lattice model in Ref.⁵⁾, for the case where the lattice points follow static worldlines. It was pointed out there that if the lattice points are static, then Killing frequency must be conserved on the lattice, so there is no possibility of a positive frequency wavepacket developing negative frequency components, which rules out any Hawking effect. This means that, on such a lattice, the in-vacuum must evolve to the Boulware vacuum at the horizon. Thus, while it is not critical to the Hawking effect that the free-fall frame be adopted, the preferred frame should not be too drastically accelerated.

If we are to entertain the possibility of a true breaking of Lorentz invariance in Nature, it seems that the asymptotic rest frame of a black hole would be the preferred one only to the extent that the black hole is at rest with respect to the cosmic preferred frame, whatever that is. As just discussed, however, it should not be important that the black hole be precisely at rest, but just that the relative boost factor γ between the black hole and the cosmic preferred frame be much smaller than k_0/κ . Only for a near-Planck mass primordial black hole in the very early universe is it conceivable that this restriction would be violated.

In a Robertson-Walker cosmology the cosmic rest frame would plausibly be the frame of the isotropic observers. In a cosmology with less symmetry, there is presumably no simple way to characterize the cutoff precisely. However, it is not implausible that in a rough approximation a preferred frame is given by the level sets of the cosmological time function^{46), 47)}, i.e. the length of the longest timelike curve back to

the initial singularity (or back to some initial slice in the quantum gravity era). Although this time function is not smooth, its first and second derivatives exist almost everywhere⁴⁶⁾, so in particular its gradient defines a local frame almost everywhere. It may be possible to construct a “phenomenological” theory using this cosmic rest frame to specify some new physics at short distances⁴⁸⁾. In such a framework, local Lorentz invariance is broken while preserving general covariance, since the preferred local frame is not “additional furniture” but rather is determined (non-locally) by the metric. The gravitational coupling of the new physics is therefore determined by its metric-dependence.

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